Supersymmetric solutions of *PT*-/non-*PT*-symmetric and non-Hermitian screened Coulomb potential via Hamiltonian hierarchy inspired variational method

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The supersymmetric solutions of *PT*-symmetric and Hermitian/non-Hermitian forms of quantum systems are obtained by solving the Schrödinger equation for the Exponential-Cosine Screened Coulomb potential. The Hamiltonian hierarchy inspired variational method is used to obtain the approximate energy eigenvalues and corresponding wave functions.

KEY WORDS: supersymmetric quantum mechanics, Hamiltonian hierarchy method, exponential-cosine screened Coulomb potential

PACS: 03.65.-w, 12.35.Jh, 21.15.-k

1. Introduction

In the past few decades, the supersymmetric approach has been profitably applied to many non-relativistic quantum mechanical systems [1–5]. The SUSYQM has provided satisfactory results concerning different non-relativistic quantum mechanical systems, such as the exactly solvable and partially solvable potentials [6–9]. The exactly solvable potentials can be understood in terms of a few basic ideas, which include supersymmetric partner potentials, shape invariance, and operator transformations.

Among the interesting problems of the non-relativistic quantum mechanics, which aim to find exact solutions to the Schrödinger equation for certain potentials of the physical interest, the screened coulomb potentials have been studied in a variety branches of physics such as atomic, nuclear, and plasma physics [10–17]. Various types of the screened Coulomb potentials like the Yukawa, Debye–Hückel and exponential-cosine screened Coulomb (ECSC) potentials are discussed in non-relativistic quantum mechanics [18–20]. While

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the screened Coulomb potential, which is in the vector coupling prescription leads to an exactly solvable for one-dimensional Dirac and Klein–Gordon equations [21–23], the Schrödinger equation for these potentials is not exactly solvable. Perturbative and approximation methods have been applied to obtain their energy eigenvalues by using hypervirial/shifted $1/N$ expansion technique, variational approach, Padé approximates, numerical integration, and group theoretical approach [24–32]. The bound state energies of some potentials like the Morse, Pöschl-Teller, and other exponential type potentials are evaluated through the SUSYQM method by following the *PT*-symmetric formalism [30–36]. *PT*-symmetric Hamiltonians satisfy the parity (*P*) and time reversal (*T*). The eigenvalue spectra of *PT*-symmetric potentials may be real or complex [37]. If *PT*-symmetry is not spontaneously broken, the form of spectra is real. For a class of non-Hermitian Hamiltonians, the concept of pseudo-hermiticity is valid [38]. In this work, the energy eigenvalues and corresponding eigenfunctions of *PT*-/non-*PT*-symmetric and non-Hermitian types of the ECSC potential are obtained by using the Hamiltonian Hierarchy method within the context of *PT*-symmetric quantum mechanics (*PT*-SQM).

This paper is organized as follows: in section 2, we give a brief pedagogical review of the Hamiltonian hierarchy method. In section 3, we apply this method for the ECSC potential. In sections 4 and 5, the method is applied for the *PT*-/non-*PT*-symmetric and non-Hermitian cases of this potential. In section 6, the results are discussed as a conclusion.

2.   Hamiltonian hierarchy method

The radial Schrödinger equation for some specific potential energies can only be solved analytically for the states with zero angular momentum [36,37]. However, in supersymmetric quantum mechanics one can deal with the hierarchy problem by using effective potentials for non-zero angular momentum states in order to solve the Schrödinger equation analytically. Hamiltonian hierarchy method suggests a hierarchy problem in the frame of the SUSYQM, in which the adjacent members are the supersymmetric partners that share the same eigenvalue spectrum except for the missing ground state.

In this method, the first step is to look for an effective potential similar to the original specific potential and inspired by the SUSYQM to propose a superpotential, namely $W_{(l+1)}(x)$, as an ansatz, where $(l+1)$ denotes the partner number with $l = 0, 1, 2 \dots$ Substituting the proposed superpotential into the Riccati equation,

$$V_{(l+1)}(x) - E^0_{(l+1)} = W^2_{(l+1)}(x) - \frac{\mathrm{d}W_{(l+1)}(x)}{\mathrm{d}x} \tag{1}$$

the $(l+1)$th member of the Hamiltonian hierarchy can be obtained. As a result, considering the shape invariance requirement [14], the bound-state energies can

be derived out through equation (1), and the corresponding eigenfunctions by means of,

$$\Psi_{(l+1)}(x) = N \exp\left(-\int^r W_{(l+1)}(x')dx'\right). \tag{2}$$

3.   Exponential-cosine screened Coulomb potential

The cosine screened Coulomb potential is written as,

$$V(r) = -\frac{q}{r}e^{-\lambda r}\cos(\mu r). \tag{3}$$

Substituting $\cos(\mu r) = (e^{i\mu r} + e^{-i\mu r})/2$ in the above potential, we get,

$$V(r) = -q\frac{e^{-\lambda r}}{r}\left(\frac{e^{i\mu r} + e^{-i\mu r}}{2}\right) \tag{4}$$

or,

$$V(r) = -\frac{q}{2}\left[\frac{e^{(i\mu-\lambda)r} + e^{-(i\mu+\lambda)r}}{r}\right]. \tag{5}$$

To simplify the calculations and for simplicity, let us take $q = 2$. Therefore,

$$V(r) = -\frac{e^{-(\lambda-i\mu)r}}{r} - \frac{e^{-(\lambda+i\mu)r}}{r}. \tag{6}$$

This potential can be considered as two separate parts as,

$$V_1(r) = -\frac{e^{-(\lambda-i\mu)r}}{r} \tag{7}$$

and,

$$V_2(r) = -\frac{e^{-(\lambda+i\mu)r}}{r}. \tag{8}$$

By defining $\lambda - i\mu = \alpha$ and $\lambda + i\mu = \beta$, the superpotential proposed as an *ansatz* for the $V_1(r)$ potential becomes,

$$W_{1(l+1)}(r) = -(l+1)\frac{\alpha e^{-\alpha r}}{1 - e^{-\alpha r}} + \frac{1}{l+1} - \frac{\alpha}{2}. \tag{9}$$

According to the Hamiltonian hierarchy method, the corresponding eigenfunction for this superpotential will be,

$$\Psi_{01}(r) = \left(1 - e^{-\alpha r}\right)^{l+1} e^{-(\frac{1}{l+1} - \frac{\alpha}{2})r}. \tag{10}$$

Assuming that the radial trial wave function is given by (10), we replace α by the variational parameter μ_1, and as a result,

$$\Psi_{\mu_1}(r) = \left(1 - e^{-\mu_1 r}\right)^{l+1} e^{-(\frac{1}{l+1} - \frac{\mu_1}{2})r}. \tag{11}$$

The variational energy is given by,

$$E_{\mu_1} = \frac{\int_0^\infty \Psi_{\mu_1}(r) \left[-\frac{1}{2}\frac{d^2}{dr^2} - \frac{e^{-\alpha r}}{r} + \frac{l(l+1)}{2r^2}\right] \Psi_{\mu_1}(r) dr}{\int_0^\infty \Psi_{\mu_1}(r)^2 dr}. \tag{12}$$

The superpotential proposed as an ansatz for the $V_2(r)$ potential is,

$$W_{2(l+1)}(r) = -(l+1)\frac{\beta e^{-\beta r}}{1 - e^{-\beta r}} + \frac{1}{l+1} - \frac{\beta}{2} \tag{13}$$

and the corresponding eigenfunction for this superpotential becomes,

$$\Psi_{02}(r) = \left(1 - e^{-\beta r}\right)^{l+1} e^{-(\frac{1}{l+1} - \frac{\beta}{2})r}. \tag{14}$$

Again, assuming that the radial trial wave function is given by (14), we can replace β by the variational parameter μ_2, and get,

$$\Psi_{\mu_2}(r) = \left(1 - e^{-\mu_2 r}\right)^{l+1} e^{-(\frac{1}{l+1} - \frac{\mu_2}{2})r}. \tag{15}$$

The variational energy is given by,

$$E_{\mu_2} = \frac{\int_0^\infty \Psi_{\mu_2}(r) \left[-\frac{1}{2}\frac{d^2}{dr^2} - \frac{e^{-\beta r}}{r} + \frac{l(l+1)}{2r^2}\right] \Psi_{\mu_2}(r) dr}{\int_0^\infty \Psi_{\mu_2}(r)^2 dr}. \tag{16}$$

Thus, by minimizing the energies E_{μ_1} and E_{μ_2} with respect to the variational parameter μ_1 and μ_2, one obtains the best estimate for the energy of the exponential-screened Coulomb potential.

As the ECSC potential is not exactly solvable, the superpotentials given by equations (9) and (13) do not satisfy the Riccati equation, but they do satisfy for effective potentials instead, $V_{1\text{eff}}$ and $V_{2\text{eff}}$ as,

$$V_{1\text{eff}}(r) = \frac{\bar{W}_1^2 - \bar{W}_1'}{2} + E(\bar{\mu}_1) \tag{17}$$

and

$$V_{2\text{eff}}(r) = \frac{\bar{W}_2^2 - \bar{W}_2'}{2} + E(\bar{\mu}_2), \tag{18}$$

where $\bar{W}_1 = W_1(\alpha = \bar{\mu}_1)$ and $\bar{W}_2 = W_2(\beta = \bar{\mu}_2)$. The $\bar{\mu}_1$ and $\bar{\mu}_2$ are the parameters that minimize the energy expectation values (12) and (16). They are given by,

$$V_{1\text{eff}}(r) = -\frac{\alpha e^{-\alpha r}}{1 - e^{-\alpha r}} + \frac{l(l+1)}{2}\frac{\alpha^2 e^{-2\alpha r}}{(1 - e^{-\alpha r})^2} + \frac{1}{2}\left(\frac{1}{l+1} - \frac{\alpha}{2}\right)^2 + E(\alpha) \quad (19)$$

and

$$V_{2\text{eff}}(r) = -\frac{\beta e^{-\beta r}}{1 - e^{-\beta r}} + \frac{l(l+1)}{2}\frac{\beta^2 e^{-2\beta r}}{(1 - e^{-\beta r})^2} + \frac{1}{2}\left(\frac{1}{l+1} - \frac{\beta}{2}\right)^2 + E(\beta). \quad (20)$$

By substituting the values of α and β in (19) and (20), one can obtain the bound state energies of the ECSC potential as,

$$E = -\frac{q}{2}\left[\frac{1}{(l+1)^2} + \frac{\lambda^2 - \mu^2}{4} - \frac{\lambda}{l+1}\right]. \quad (21)$$

It is interesting to notice that for $\mu = 0$, the ECSC potential reduces to the to the form called Yukawa potential and that equation (21) reduces to,

$$E = -\frac{q}{2}\left[\frac{1}{(l+1)} - \frac{\lambda}{2}\right]^2. \quad (22)$$

4.   Non-PT-symmetric and non-Hermitian exponential-cosine screened Coulomb case

The non-PT and non-Hermitian cosine screened Coulomb potential can be defined as,

$$V(r) = \frac{iq}{r}e^{-\lambda r}\cos(\mu r) \quad (23)$$

or simply,

$$V(r) = \frac{iq}{2}\left(\frac{e^{-\alpha r}}{r} + \frac{e^{-\beta r}}{r}\right). \quad (24)$$

In this case the proposed superpotentials can be,

$$W_{1(l+1)}(r) = -(l+1)\frac{i\alpha e^{-\alpha r}}{1 - e^{-\alpha r}} + \frac{1}{l+1} - \frac{\alpha}{2} \quad (25)$$

and

$$W_{2(l+1)}(r) = -(l+1)\frac{i\beta e^{-\beta r}}{1 - e^{-\beta r}} + \frac{1}{l+1} - \frac{\beta}{2}. \quad (26)$$

Although the superpotentials are complex, following the same method will yield the same energy eigenvalues as in (21).

5.   *PT*-symmetric and non-Hermitian exponential-cosine screened Coulomb case

The *PT*-symmetric and non-Hermitian cosine screened Coulomb potential can be introduced as,

$$V(r) = -\frac{q}{r}e^{-i\lambda r}\cos(\mu r) \tag{27}$$

or,

$$V(r) = -\frac{q}{2}\left[\frac{e^{-i(\lambda-\mu)r} + e^{-i(\lambda+\mu)r}}{r}\right]. \tag{28}$$

Taking, $(\lambda - \mu) = -\alpha_0$ and $\lambda + \mu = \beta_0$, we will have,

$$V(r) = -\frac{q}{2}\left(\frac{e^{-i\alpha_0 r}}{r} + \frac{e^{-i\beta_0 r}}{r}\right) \tag{29}$$

and as a result the superpotentials can be proposed as,

$$W_{1(l+1)}(r) = -(l+1)\frac{\alpha_0 e^{-i\alpha_0 r}}{1 - e^{-i\alpha_0 r}} + \frac{1}{l+1} - \frac{\alpha_0}{2} \tag{30}$$

and

$$W_{2(l+1)}(r) = -(l+1)\frac{\beta_0 e^{-i\beta_0 r}}{1 - e^{-i\beta_0 r}} + \frac{1}{l+1} - \frac{\beta_0}{2}. \tag{31}$$

In conclusion, by applying the method the same energy eigenvalues will be obtained as in (21).

6.   Conclusions and remarks

We have applied the Hamiltonian hierarchy method within the framework of the SUSYQM formulation by presenting a superpotential that yields a trial function to calculate the approximate bound state energies and corresponding eigenfunctions for the ECSC potential. We have also considered its different symmetric forms in our calculations. As the energy spectrum of the *PT*-invariant complex-valued non-Hermitian potentials may be real or complex depending on the parameters, we have clarified that there are some restrictions on the potential parameters for the bound states in *PT*-symmetric, or more generally, in non-Hermitian quantum mechanics. Furthermore, it is shown that the superpotentials, their superpartners and the corresponding ground state eigenfunctions satisfy the *PT*-symmetry condition.

Finally, we can add that our approximate yet accurate results of complexified ECSC potential by the justification of the numerical results presented in

Table 1
Energy eigenvalues of ECSC as a function of the screening parameter λ for 1*s*, 2*p*, 3*d*, and 4*f* states in Rydberg units of energy.

Screening λ	SUSYQM our work	Hypervirial solution [40]	NR–QM variational [20]	Exact numerical [39]
State 1*s*				
0.020	−0.480290	−0.480310	−0.480300	−0.480300
0.050	−0.451810	−0.451800	−0.451820	−0.451800
0.080	−0.424560	−0.424560	−0.424570	–
0.100	−0.407070	−0.407050	−0.470600	−0.407100
State 2*p*				
0.020	−0.211800	−0.105890	−0.211900	−0.211900
0.050	−0.162500	−0.080400	−0.161500	–
0.080	−0.050500	−0.046000	–	–
0.100	−0.092860	−0.008000	−0.092890	−0.093070
State 3*d*				
0.020	−0.075020	−0.037500	−0.075030	−0.075030
0.050	−0.033620	−0.017340	−0.033740	−0.033830
0.080	−0.009020	−0.008000	–	–
0.100	−0.038889	–	–	–
State 4*f*				
0.020	−0.028750	−0.014700	−0.028970	–
0.050	−0.004100	−0.003200	–	–
0.080	−0.184500	−0.175000	–	–
0.100	−0.018700	–	–	–

table 1 motivate an appropriate approach to analyze the exactly and non-exactly solvable potentials. We believe that this method may increase the number of applications in the study of different quantum systems.

Acknowledgments

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